Int. J. Heat Mass Transfer. Vol. 28, No. 6, pp. 1089–1095, 1985
Printed in Great Britain

0017–9310/85 $3.00 + 0.00

A potential and stream function analysis of two-dimensional steady-state convective diffusion equations involving Laplace fields

P. K. WANG

Department of Meteorology, University of Wisconsin, Madison, WI 53706, U.S.A.

(*Received* 6 *March* 1984 *and in final form* 4 *December* 1984)

Abstract—A special technique in solving steady-state two-dimensional convective diffusion equations involving external force fields having Laplace potentials is presented. The technique uses the fact that it is possible to find a conjugate stream function to the potential and the two can thus be considered as independent coordinates. The convective diffusion equation can be transformed into the potential and stream function coordinates and the resulting equation is separable in term of these two new coordinates. Two examples are illustrated. A pseudo-two-dimensional problem is also presented to show the usefulness of the technique even when not all requirements are met. Relations with Helmholtz and Schrödinger equations are discussed.

1. INTRODUCTION

THE MASS transfer phenomena in physics and engineering can often be described by the diffusion equations. The heat transfer problems are usually described by the heat conduction equations which have similar forms as diffusion equations. Therefore it is sufficient to discuss only the diffusion equation in this paper with the understanding that the results will also be applicable to heat transfer problems. When the diffusion processes occur in the presence of external force fields they are regarded as convective diffusion. We want to address the convective diffusion in the steady state here. Thus in the presence of external forces, the steady-state convective diffusion equation (sometimes called the Fokker–Planck Equation) is in the following form (cf. refs. [1] and [2]):

$$D\nabla^2 n - \nabla\cdot(\mathbf{F}n) = 0 \tag{1}$$

where n represents the concentration, D the diffusion coefficient, $\mathbf{F}$ the net external force. In writing equation (1) we have assumed that the diffusion coefficient is a constant and will be neglected in the following discussions as it is unimportant mathematically. We are also going to treat other quantities as dimensionless for the sake of lucidity.

In this paper a special class of force fields will be examined, namely, the conservative force fields. Such forces satisfy the irrotational condition

$$\nabla\times\mathbf{F} = 0 \tag{2}$$

and therefore there exists a scalar potential ϕ such that

$$\mathbf{F} = -\nabla\phi. \tag{3}$$

We will further restrict our attention to the potential fields that satisfy the Laplace equation:

$$\nabla^2\phi = 0. \tag{4}$$

From the point of view of equation (3) this is equivalent to saying that force fields are non-divergent.

The restriction on the present method is that the flow be two-dimensional (2-D), irrotational, and incompressible. This requires that the diffused substance (heat, salt, etc.) be a passive scalar and that this passive scalar diffuses faster than momentum. Thus it appears that the method might apply in high Reynolds number, order one Peclet number forced convective flow (e.g. flow at low Prandtl number). Despite seemingly very restrictive conditions, such potential fields (and hence the corresponding forces) occur quite frequently in physical problems. Common examples include irrotational flow of a perfect fluid, surface waves, electromagnetic phenomena, and gravitation. All these external forces can be imposed on a pure diffusion system to form convective diffusion system. Convective diffusion problems involving these forces may have been considered separately before but they are not considered as a special group. When this special property is recognized, these problems can be thought of as a special class and their properties can be studied more systematically. It is the purpose of the present paper to examine the solutions of these problems in two-dimensional form employing a special technique, namely, the potential and stream function analysis. As will be seen later, this technique is especially useful when the equation cannot be separated in its original coordinates.

2. POTENTIAL AND STREAM FUNCTIONS

It is well-known in fluid dynamics that in incompressible, irrotational flow the fluid velocity is non-divergent and can be derived from a scalar potential ϕ which satisfies equation (4). If, in addition, the flow is two-dimensional, then there exists a stream function ψ conjugate to ϕ such that (cf. [3])

$$\frac{\partial\psi}{\partial x_1} = \frac{\partial\phi}{\partial x_2}, \qquad \frac{\partial\psi}{\partial x_2} = -\frac{\partial\phi}{\partial x_1} \tag{5}$$

NOMENCLATURE

D	diffusion coefficient
$\mathbf{F}$	net external forces
f	arbitrary scalar function
K	thermal conductivity
n	concentration of particles
T	temperature
q	particle charges
r	radial distance in spherical coordinate system
x, y, z	Cartesian coordinates
x_1, x_2	any generalized two-dimensional coordinates.

Greek symbols

ϕ	potential function of the external force
ψ	stream function conjugate to ϕ
θ	zenith angle in spherical coordinate system.

where x_1 and x_2 are two generalized coordinates. Obviously ψ also satisfies the Laplace equation (4) and thus can be considered as a potential function itself. It is therefore possible to form a complex potential $u = \phi + i\psi$ and the related problems can be treated using complex analysis. In the present paper, however, we will not consider the complex method but regard ϕ and ψ as two separate real functions.

From equations (4) and (5) we immediately have the following relations:

$$\nabla^2\phi = \nabla^2\psi = 0, \tag{6}$$

$$\nabla\phi \cdot \nabla\psi = 0, \tag{7}$$

$$(\nabla\phi)^2 = (\nabla\psi)^2, \tag{8}$$

where $(\nabla\phi)^2$ means $\nabla\phi \cdot \nabla\phi$. All the operations are in two dimensions. The relation (7) merely says that the ϕ- and ψ-surface are orthogonal to each other. Due to this orthogonality it is possible to treat these two functions as independent orthogonal coordinates as has been mentioned in ref. [4]. Now since both ϕ and ψ are scalar fields, if a function f is such that $f = f[\phi(x_1, x_2), \psi(x_1, x_2)]$, then the following differential operations of f are valid:

$$\nabla f = \frac{\partial f}{\partial \phi}\nabla\phi + \frac{\partial f}{\partial \psi}\nabla\psi, \tag{9}$$

$$\begin{aligned}\nabla^2 f &= \frac{\partial^2 f}{\partial \phi^2}(\nabla\phi)^2 + \frac{\partial f}{\partial \phi}(\nabla^2\phi) \\ &\quad + \frac{\partial^2 f}{\partial \psi^2}(\nabla\psi)^2 + \frac{\partial f}{\partial \psi}(\nabla^2\psi) \\ &= \frac{\partial^2 f}{\partial \phi^2}(\nabla\phi)^2 + \frac{\partial^2 f}{\partial \psi^2}(\nabla\psi)^2.\end{aligned} \tag{10}$$

Although these operations are based on the assumption that the potential fields are Laplacian, it is to be remembered that a stream function ψ can be defined when the potential is not Laplacian and even when there is no potential at all. However in these latter cases, relations (6)–(9) do not hold simultaneously. Nevertheless the present transformational technique sometimes can also be applied to these problems and obtain useful results as will be shown in Section 6.

3. THE TRANSFORMED 2-D STEADY-STATE CONVECTIVE DIFFUSION EQUATION AND SOLUTIONS

Now let's consider equation (1) under the assumption that the force potential is a Laplace field. Equation (1) then becomes

$$\nabla^2 n + \nabla\phi \cdot \nabla n = 0. \tag{11}$$

Transforming equation (11) to (ϕ, ψ)-coordinates using relations (9) and (10), we obtain

$$\frac{\partial^2 n}{\partial \phi^2}(\nabla\phi)^2 + \frac{\partial^2 n}{\partial \psi^2}(\nabla\psi)^2 + \nabla\phi \cdot \left(\frac{\partial n}{\partial \phi}\nabla\phi + \frac{\partial n}{\partial \psi}\nabla\psi\right) = 0. \tag{12}$$

Using relations (6)–(8), equation (12) becomes

$$\frac{\partial^2 n}{\partial \phi^2} + \frac{\partial^2 n}{\partial \psi^2} + \frac{\partial n}{\partial \phi} = 0 \tag{13}$$

which is an extremely simple equation and the conventional separation-of-variables method is readily applicable. In writing (13) we have assumed $(\nabla\psi)^2 \neq 0$, $(\nabla\phi)^2 \neq 0$. Thus letting

$$n = \Phi(\phi)\Psi(\psi) \tag{14}$$

and substituting into (13), we immediately obtain the two separated equations

$$\begin{cases} \dfrac{\partial^2 \Phi}{\partial \phi^2} + \dfrac{\partial \Phi}{\partial \phi} - \lambda^2 \Phi = 0 & (15) \\[2ex] \dfrac{\partial^2 \Psi}{\partial \psi^2} + \lambda^2 \Psi = 0 & (16) \end{cases}$$

where λ^2 is the separation constant which can be any number. The solution of (15) is

$$\Phi = \begin{cases} C_1(\lambda)\exp\left(\dfrac{-1+\sqrt{1+4\lambda^2}}{2}\phi\right) \\[2ex] +C_2(\lambda)\exp\left(\dfrac{-1-\sqrt{1+4\lambda^2}}{2}\phi\right), \quad (\lambda \neq 0) & (17) \\[2ex] a_0 + b_0\exp(-\phi), \quad (\lambda = 0) & (18) \end{cases}$$

and the solution of (16) is

$$\Psi = \begin{cases} C_3(\lambda)\cos\lambda\psi + C_4(\lambda)\sin\lambda\psi, & (\lambda \neq 0) \quad (19) \\ c_0 + d_0\psi, & (\lambda = 0) \quad (20) \end{cases}$$

where a_0, b_0, c_0, and d_0 are arbitrary constants, and C_1, C_2, C_3, and C_4 are coefficients which are only a function of the separation constant λ. The most general solution is therefore

$$n = a_0 + b_0\exp(-\phi) + [c_0 + d_0\exp(-\phi)]\psi + \int_{-\infty}^{\infty}\{[C_1(\lambda)\exp(\alpha\phi) + C_2(\lambda)\exp(\beta\phi)]\cos\lambda\psi + [C_3(\lambda)\exp(\alpha\phi) + C_4(\lambda)\exp(\beta\phi)]\sin\lambda\psi\}\,d\lambda \quad (21)$$

where

$$\alpha = \frac{-1+\sqrt{1+4\lambda^2}}{2}, \quad \beta = \frac{-1-\sqrt{1+4\lambda^2}}{2}. \quad (22)$$

It may be possible to absorb every term into the integral but the form presented here is probably easier to work with. Equation (21) looks very much alike the conventional solutions of standard diffusion equations except it is expressed as a function of the potential ϕ and stream function ψ. In determining the constants, of course, one has to apply boundary conditions and usually we have to express ϕ and ψ in the original physical coordinates. But in many cases of the diffusion problems involving Laplace potential fields one or more of the physical boundaries coincide with the potential or stream surfaces. When this is the case the determination of the constants becomes particularly simple. Note that (17) is for real λ. The case of imaginary λ can be written analogously.

4. A SIMPLE EXAMPLE IN CARTESIAN COORDINATES

In order to illustrate the use of this technique, let us consider an actual example. Suppose we wish to determine the steady-state concentration profile of charged particles in the vicinity of two semi-infinite but mutually perpendicular conducting walls. The particles each carry charge q and the walls are equipotential at a value V_0. Assuming that the medium in which particles reside is stagnant and the gravitational settling is negligible, then the diffusion equation governing the particle concentration is

$$D\nabla^2 n + q\nabla V\cdot\nabla n = 0 \quad (23)$$

where V is the electric potential produced by the two charged walls. Neglecting the space charge effect due to particles, the potential distribution in the present case is obviously

$$V(x,y) = V_0 + 2xy \quad (24)$$

which are hyperbolas. Equation (23) in (x,y)-coordinates will be

$$D\left(\frac{\partial^2 n}{\partial x^2} + \frac{\partial^2 n}{\partial y^2}\right) + 2q\left(y\frac{\partial n}{\partial x} + x\frac{\partial n}{\partial y}\right) = 0 \quad (25)$$

equation (25) cannot be separated into (x,y)-coordinates. However it can be easily shown that the potential function $\phi(=qV)$ and its conjugate stream function $\psi[=q(x^2-y^2)]$ satisfy the relations (6)–(8). The complete solution is therefore represented by equation (21).

Now if we impose the boundary conditions such that the concentration faraway from the walls is a constant n_∞ and the concentration on the walls is zero, i.e. the walls are perfect sinks of particles, then we have

$$\begin{cases} n = 0 \text{ at } x = 0 \text{ or } y = 0 & (\text{i.e. } xy = 0) \quad (26) \\ n = n_\infty \text{ at } x \to \infty \text{ and } y \to \infty & (\text{i.e. } xy \to \infty). \quad (27) \end{cases}$$

It is seen that this pair of conditions is equivalent to

$$n = 0 \quad \text{at } \phi = qV_0 \quad (28)$$

$$n = n_\infty \text{ at } \phi \to \infty \quad (29)$$

and the solution which satisfies these conditions is simply the first two terms of (21):

$$n = n_\infty(1 - e^{-2xy}). \quad (30)$$

Thus the present transformation method provides an easy answer to the otherwise non-separable differential equation. Of course, more complicated boundary conditions can be prescribed and more complicated solutions will result. However it is not the purpose of the present paper to examine these complicated cases. In the following discussions we will not prescribe boundary conditions.

5. A SECOND EXAMPLE

Consider the heat transfer problem in which an infinite circular cylinder of unit radius is surrounded by a fluid in potential flow with unit free stream velocity. If steady state is reached, then the equation governing this situation is

$$K\nabla^2 T - \mathbf{V}\cdot\nabla T = 0 \quad (31)$$

where K is the thermal conductivity of the fluid and $\mathbf{V}$ is the flow velocity. The potential and stream functions in this case are

$$\phi = \left(r + \frac{1}{r}\right)\cos\theta, \qquad \psi = \left(r - \frac{1}{r}\right)\sin\theta, \quad (32)$$

respectively. The boundary conditions are

$$T = T_1(\theta) \text{ at } r = 1, \qquad T = T_\infty \quad \text{at } r \to \infty. \quad (33)$$

Since (31) is of the same form as (11), the solution is given by (21). That solution has to satisfy the boundary conditions (33). Only the upstream region will be examined here. The solution of the downstream region can be determined in a similar way and hence need not be elaborated here.

In the upstream region we have $\cos\theta > 0$. Since T is finite, all terms that would grow infinitely with r in (21) should be discarded. Therefore we set $C_1(\lambda) = 0$, $c_0 = d_0 = 0$. We can also set $C_3(\lambda) = C_4(\lambda) = 0$ in this case. The b_0 term can be reproduced by setting $\lambda = 0$ in the integral and therefore can also be eliminated here. With all these considerations, the solution becomes

$$T = a_0 + \int_{-\infty}^{\infty} C(\lambda) \times \exp\left[\frac{-(1+\sqrt{1+4\lambda^2})}{2}\frac{\phi}{K}\right]\cos\lambda\left(\frac{\psi}{K}\right)\mathrm{d}\lambda. \quad (34)$$

Equation (33) then requires that

$$T_1(\theta) = a_0 + \int_{-\infty}^{\infty} C(\lambda) \times \exp\left[-\frac{(1+\sqrt{1+4\lambda^2}}{2}\frac{2\cos\theta}{K}\right]\mathrm{d}\lambda \quad (\because \psi_{r=1} = 0) \quad (35)$$

and

$$T_\infty = a_0 + \int_{-\infty}^{\infty} C(\lambda)\exp\left[-\frac{(1+\sqrt{1+r\lambda^2})}{2}\frac{r\cos\theta}{K}\right] \times \cos\lambda(r\sin\theta)\,\mathrm{d}\lambda|_{r\to\infty} = a_0 \quad (36)$$

since the exponential function is zero in the $r \to \infty$ (35) now becomes

$$T_1(\theta) - T_\infty = \int_{-\infty}^{\infty} C(\lambda) \times \exp\left[-\frac{(1+\sqrt{1+4\lambda^2})}{K}\cos\theta\right]\mathrm{d}\lambda. \quad (37)$$

Integral transformation techniques will be useful in determining $C(\lambda)$ here. Under certain conditions $C(\lambda)$ can be easily determined. For example in many cases $C(\lambda)$ is an even function of λ. Then we let

$$m+2 = 1+\sqrt{1+4\lambda^2}. \quad (38)$$

$$\lambda = \frac{\sqrt{m^2+2m}}{2} \text{ and } \mathrm{d}\lambda = \frac{(m+1)}{2\sqrt{m^2+2m}}\mathrm{d}m. \quad (39)$$

Equation (37) can now be written

$$T_1(\theta) - T_\infty = 2\int_0^\infty \frac{(m+1)C(m)}{2\sqrt{m^2+2m}} \mathrm{e}^{-\alpha(m+2)}\,\mathrm{d}m \quad (40)$$

where

$$\alpha = \cos\theta/K. \quad (41)$$

Equation (40) can be written as

$$T(\alpha) = [T_1(\theta) - T_\infty]\,\mathrm{e}^{2\alpha} = \int_0^\infty D(m)\,\mathrm{e}^{-\alpha m}\,\mathrm{d}m \quad (42)$$

where

$$D(m) = \frac{(m+1)C(m)}{\sqrt{m^2+2m}}. \quad (43)$$

Equation (42) is clearly a standard form of the Laplace transform and therefore $D(m)$ can be determined by the Bromwich integral

$$D(m) = \frac{1}{2\pi\mathrm{i}}\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} \mathrm{e}^{\alpha m} T(\alpha)\,\mathrm{d}\alpha \quad (44)$$

(see, e.g. ref. [4]). This determines the exact solution (34) once $T_1(\theta)$ and T_∞ are explicitly specified. For example, if the upstream surface temperature of the cylinder is given by

$$T_1(\theta) - T_\infty = \frac{1}{(2+\cos\theta)\exp(2\cos\theta)}, \quad (45)$$

then we can determine $D(m)$ from (44) and then convert to $C(\lambda)$ which is

$$C(\lambda) = \sqrt{1 - \frac{1}{1+4\lambda^2}}\exp(2 - 2\sqrt{1+4\lambda^2})$$

and the temperature distribution in the fluid is given by

$$T(r,\theta) = T_\infty + 2\int_0^\infty \left\{\sqrt{1-\frac{1}{1+4\lambda^2}}\exp(2-2\sqrt{1+4\lambda^2}) \times \exp\left[-\frac{(1+\sqrt{1+4\lambda^2})}{2}\left(r+\frac{1}{r}\right)\frac{\cos\theta}{K}\right] \times \cos\left[\lambda\left(r-\frac{1}{r}\right)\frac{\sin\theta}{K}\right]\right\}\mathrm{d}\lambda. \quad (46)$$

This can be readily integrated to give $T(r, \theta)$ at any (r, θ)-point. More extensive applications of this method of solution will be considered in the future.

6. A PSEUDO-2D PROBLEM

The successful application of the present method depends on the special properties of the external force fields involved. If they are such that relations (6)–(8) are satisfied, then the solution is given by equation (21). However, sometimes even if equations (6)–(8) are not completely valid, applying the present method may reduce the difficulty in solving the original equation and obtaining useful results, as is illustrated in the following example.

Let us consider a spherical diffusion system in which the associated external force is not spherically symmetric. If the force can be derived from a potential, the potential function will not be radially symmetric either. The simplest example is

$$\phi = r\cos\theta \quad (47)$$

where r is the radial distance and θ is the zenith angle of the spherical coordinate system. Obviously this potential corresponds to a force that is uniform in the z-direction (e.g. gravitational force). Assuming there is an azimuthal symmetry so that only r- and θ-coordinates are involved in the problem. Note that a spherical system is not a 2-D system even though only two coordinates appear in the equation. Nevertheless we see that

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial(r\cos\theta)}{\partial r}\right] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial(r\cos\theta)}{\partial\theta}\right] = \frac{2}{r}\cos\theta - \frac{2}{r}\cos\theta = 0. \tag{48}$$

It is also possible to find a conjugate stream function ψ:

$$\psi = \ln r|\sin\theta|. \tag{49}$$

Since

$$\frac{\partial\phi}{\partial r} = \cos\theta,\quad \frac{\partial\psi}{\partial r} = \frac{1}{r},\qquad \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\sin\theta,\quad \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{\cot\theta}{r}, \tag{50}$$

we easily see that $\nabla^2\psi = 0$ and $\nabla\psi\cdot\nabla\phi = 0$, but

$$\frac{\csc^2\theta}{r^2} = (\nabla\psi)^2 \neq (\nabla\phi)^2 = 1. \tag{51}$$

Thus condition (8) is violated and the solution (21) cannot be applied. However we can still proceed the transformation to (ϕ, ψ)-coordinates and hope that the equation can be reduced in a manageable form. Now the convective diffusion equation in front of us is, in spherical coordinates,

$$\nabla^2 n + \nabla\phi\cdot\nabla n = \frac{\partial^2 n}{\partial r^2} + \frac{2}{r}\frac{\partial n}{\partial r} + \frac{1}{r^2}\frac{\partial^2 n}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial n}{\partial\theta} + \cos\theta\frac{\partial n}{\partial r} - \sin\theta\frac{\partial n}{\partial\theta} = 0. \tag{52}$$

This is again an equation that cannot be separated into r- and θ-parts. Now proceed with the (ϕ, ψ)-transformation. We get

$$\nabla^2 n + \nabla\phi\cdot\nabla n = \frac{\partial^2 n}{\partial\phi^2}(\nabla\phi)^2 + \frac{\partial^2 n}{\partial\psi^2}(\nabla\psi)^2 + \frac{\partial n}{\partial\phi}(\nabla\phi)^2 = 0 \tag{53}$$

using the fact that $\nabla^2\psi = \nabla^2\phi = 0$ and $\nabla\psi\cdot\nabla\phi = 0$. Using equation (51), we have

$$\frac{\partial^2 n}{\partial\phi^2} + \frac{\partial n}{\partial\phi} + \frac{\partial^2 n}{\partial\psi^2}\left(\frac{1}{r^2\sin^2\theta}\right) = 0. \tag{54}$$

The term in the bracket is

$$\left(\frac{1}{r^2\sin^2\theta}\right) = e^{-2\psi} \tag{55}$$

and therefore equation (54) becomes

$$\frac{\partial^2 n}{\partial\phi^2} + \frac{\partial n}{\partial\phi} + \frac{\partial^2 n}{\partial\psi^2}e^{-2\psi} = 0. \tag{56}$$

Again letting $n = \Phi(\phi)\Psi(\psi)$, we obtain the two separated equations

$$\begin{cases} \dfrac{\partial^2\Phi}{\partial\phi^2} + \dfrac{\partial\Phi}{\partial\phi} - \lambda^2\Phi = 0 & (57) \\[2ex] \dfrac{\partial^2\Psi}{\partial\psi^2} + \lambda^2 e^{2\psi}\Psi = 0 & (58) \end{cases}$$

where λ^2 is again the separation constant. Equation (57) is the same as (15) and therefore the solution is given by

$$\Phi = C_1(\lambda)\exp\left(\frac{-1+\sqrt{1+4\lambda^2}}{2}\phi\right) + C_2(\lambda)\exp\left(\frac{-1-\sqrt{1+4\lambda^2}}{2}\phi\right). \tag{59}$$

On the other hand equation (42) looks like the time-independent Schrödinger equation with a logarithmic potential and the standard way to proceed is (cf. [4]) by changing the independent variable ψ to ξ by letting

$$\xi = \lambda e^{\psi}. \tag{60}$$

Then

$$\begin{aligned} \frac{\partial^2\Psi}{\partial\psi^2} &= \frac{\partial}{\partial\psi}\left(\frac{\partial\Psi}{\partial\xi}\frac{\partial\xi}{\partial\psi}\right) = \frac{\partial}{\partial\psi}\left(\lambda e^{\psi}\frac{\partial\Psi}{\partial\psi}\right) \\ &= \frac{\partial}{\partial\psi}\left(\xi\frac{\partial\Psi}{\partial\xi}\right) = \frac{\partial\xi}{\partial\psi}\frac{\partial\Psi}{\partial\xi} + \xi\frac{\partial}{\partial\psi}\left(\frac{\partial\Psi}{\partial\xi}\right) \\ &= \xi\frac{\partial\Psi}{\partial\xi} + \xi\frac{\partial}{\partial\xi}\left(\frac{\partial\Psi}{\partial\psi}\right) = \xi\frac{\partial\Psi}{\partial\xi} + \xi^2\frac{\partial^2\Psi}{\partial\xi^2}. \end{aligned} \tag{61}$$

Putting equation (61) into equation (58), we have

$$\xi^2\frac{\partial^2\Psi}{\partial\xi^2} + \xi\frac{\partial\Psi}{\partial\xi} + \xi^2\Psi = 0 \tag{62}$$

or

$$\frac{\partial^2\Psi}{\partial\xi^2} + \frac{1}{\xi}\frac{\partial\Psi}{\partial\xi} + \Psi = 0 \tag{63}$$

with the caution that now $\lambda \neq 0$ (so that $\xi^2 \neq 0$). On the other hand $e^{2\psi}$ is always positive since $\psi \geqslant 0$. The solution of equation (63) are the well-known Bessel functions of zeroth order, i.e.

$$\Psi(\xi) = C_3(\lambda)J_0(\xi) + C_4(\lambda)Y_0(\xi) \tag{64}$$

or

$$\Psi(\psi) = C_3(\lambda)J_0(\lambda e^{\psi}) + C_4(\lambda)Y_0(\lambda e^{\psi}),\quad (\lambda \neq 0) \tag{65}$$

where J_0 and Y_0 are the zeroth-order Bessel function of first kind and second kind, respectively. When $\lambda = 0$, we have

$$\Psi(\psi) = a_0 + b_0\psi. \tag{66}$$

The general solution of the present diffusion equation is therefore

$$n = a_0 + b_0\psi + \int_{-\infty}^{\infty} \{[C_1(\lambda)\exp(\alpha\phi) + C_2(\lambda)\exp(\beta\phi)]J_0(\lambda e^{\psi}) + [C_3(\lambda)\exp(\alpha\phi) + C_4(\lambda)\exp(\beta\phi)]\,Y_0(\lambda e^{\psi})]\}\,d\lambda \tag{67}$$

where α and β are given by equation (22).

It is therefore seen here that even when equations (6)–(8) are not completely satisfied, the potential and stream function analysis can reduce the difficulty of the problem solving and useful solutions may be obtained. Of course, various constants in equation (67) are to be fixed by boundary conditions.

6. RELATIONS WITH OTHER EQUATIONS

It is quite clear that the convective diffusion equation

$$\nabla^2 n - \nabla\phi \cdot \nabla n = 0 \tag{68}$$

differs from equation (11) only by the sign of the second term. It is therefore only necessary to set a new potential function $\phi' = -\phi$ and then proceed in exactly the same way as before.

An interesting property of equation (11) is that it can be transformed to the time independent Schrödinger equation via the change of dependent variable

$$n = n^* \exp\left(-\frac{\phi}{2}\right). \tag{69}$$

The transformed equation becomes

$$\nabla^2 n^* - \frac{1}{4}(\nabla\phi)^2 n^* = 0. \tag{70}$$

Equation (52) can be transformed using the same method and becomes the Helmholtz equation

$$\nabla^2 n^* + \frac{1}{4}(\nabla\phi)^2 n^* = 0. \tag{71}$$

All these are possible because $\nabla^2\phi = 0$. This transformation has been used in refs. [5–7]. Note that these transformations are valid in general, not just limited to two-dimensional cases. There exist particular solutions for all these equations, namely,

$$n = \begin{cases} A\exp(-\phi) + B, & \text{for equation (11)} \\ A\exp(\phi) + B, & \text{for equation (68)} \end{cases} \tag{72}$$

and

$$n^* = \begin{cases} A\exp(\phi/2) + B\exp(-\phi/2), & \text{for equation (70)} \\ A\exp(i\phi/2) + B\exp(-i\phi/2), & \text{for equation (71).} \end{cases}$$

Again these particular solutions are valid in general and not limited to the two-dimensional form. They can be verified easily by simply substituting back to the original equations.

Naturally, equations (70) and (71) in two dimensions can again be treated by the potential and stream function analysis. For example, after the (ϕ, ψ)-transformation, equation (70) becomes

$$\frac{\partial^2 n^*}{\partial\phi^2} + \frac{\partial^2 n^*}{\partial\psi^2} - \frac{1}{4}n^* = 0 \tag{73}$$

and the separated equations are

$$\begin{cases} \dfrac{\partial^2\Phi}{\partial\phi^2} - \lambda^2\phi = 0 \\ \dfrac{\partial^2\Psi}{\partial\psi^2} - \left(\dfrac{1}{4} - \lambda^2\right)\Psi = 0 \end{cases} \tag{74}$$

and the general solution is therefore

$$n = \int_{-\infty}^{\infty} \left\{\left[C_1(\lambda)\exp\left(\sqrt{\frac{1}{4} - \lambda^2}\psi\right) + C_2(\lambda)\exp\left(-\sqrt{\frac{1}{4} - \lambda^2}\psi\right)\right]\exp(\lambda\phi) + \left[C_3(\lambda)\exp\left(\sqrt{\frac{1}{4} - \lambda^2}\psi\right) + C_4(\lambda)\exp\left(-\sqrt{\frac{1}{4} - \lambda^2}\psi\right)\right]\exp(-\lambda\phi)\right\}d\lambda. \tag{75}$$

The solution of equation (71) is similar.

7. CONCLUSIONS

It is demonstrated in previous sections that the steady-state two-dimensional convective diffusion equations involving conservative external forces which have Laplace potentials can be solved by transforming to the potential and stream function, or (ϕ, ψ)-coordinates. The resulting equation is readily separable and solutions easily obtained. Sometimes even when the system is not truly two-dimensional, the method can still be useful in obtaining solutions. These results apply to the heat transfer problems as well. It is also seen that there are related Helmholtz and Schrödinger equations that can be solved similarly. These solutions may have some applications in scattering problems. Since the Laplace-type potential fields are frequently encountered in many physical and engineering problems, it is expected that the present method should have useful applications. Also since the formulation in Section 3 does not depend on any particular coordinate systems, the method probably will be very useful in treating 2-D problems with irregular boundary surfaces, provided that one can determine the ϕ- and ψ-functions. Various applications are currently under investigation by us and will be reported in the near future.

Acknowledgements—I would like to thank two anonymous reviewers whose suggestions strengthened this paper. I also thank Mrs Eva Singer who carefully typed the manuscript. This work is performed under the partial support of NSF Grants ATM-8317602 and 8120809 to the University of Wisconsin at Madison.

REFERENCES

1. W. Jost, *Diffusion in Solids, Liquids, and Gases*. Academic Press, New York (1960).
2. R. B. Bird, W. E. Stewart and E. L. Lightfoot, *Transport Phenomena*. John Wiley, New York (1960).
3. H. Lamb, *Hydrodynamics*. Dover, New York (1945).
4. P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*, Vols. I and II. McGraw-Hill, New York (1953).
5. N. Byers and C. N. Yang, Theoretical considerations concerning quantized magnetic flux in superconducting cylinders, *Phys. Rev. Lett.* **7**, 46–49 (1961).
6. P. K. Wang, A theoretical and experimental study on the scavenging of aerosol particles by water drops. Ph.D. thesis, Dept. of Atmospheric Science, University of California—Los Angeles (1978).
7. P. K. Wang, Collection of aerosol particles by a conducting sphere in an external electric field—continuum regime approximation, *J. Colloid Interf. Sci.* **94**, 301–318 (1983).

ANALYSE DES EQUATIONS DE CONVECTION STATIONNAIRE BIDIMENSIONNELLE A CHAMP LAPLACIEN, RAPPORTEES AUX FONCTIONS POTENTIELLES ET DE COURANT

Résumé—On présente une technique spéciale de résolution de l'équation de convection permanente et bidimensionnelle, pour des champs de force ayant des potentiels de Laplace. On utilise le fait qu'il est possible de trouver une fonction de courant conjuguée au potentiel et les deux peuvent être considérés comme des coordonnées indépendantes. L'équation de convection peut être transformée dans les coordonnées de fonction potentielle et de courant et l'équation résultante est séparable en fonction de ces deux nouvelles coordonnées. Deux exemples sont illustrés. Un pseudo problème bidimensionnel est présenté pour montrer l'utilité de la technique même si toutes les exigences ne sont pas remplies. On discute des relations avec les équations d'Helmoltz et de Schrödinger.

UNTERSUCHUNG DER ZWEIDIMENSIONALEN STATIONÄREN KONVEKTIVEN TRANSPORTGLEICHUNGEN MITTLES POTENTIAL- UND STROMFUNKTION UNTER EINBEZUG VON LAPLACE-FELDERN

Zusammenfassung—Es wird eine spezielle Methode zur Lösung der stationären zweidimensionalen konvektiven Transportgleichung vorgestellt. Sie verwendet äußere Kräftefelder mit Laplace-Potentialen. Die Methode nutzt aus, daß es möglich ist, eine zum Potential konjugierte Stromfunktion zu finden, und daß beide somit als unabhängige Koordinaten betrachtet werden können. Die konvektive Transportgleichung kann in die Koordinaten der Potential- und Stromfunktion überführt werden, und die resultierende Gleichung ist mit diesen beiden neuen Koordinaten separierbar. Zwei Beispiele werden dargestellt. Ein Pseudo-2D-Problem wird dargestellt, um den Nutzen der Methode zu zeigen, selbst wenn nicht alle Voraussetzungen erfüllt sind. Zusammenhänge mit den Gleichungen von Helmholtz und Schrödinger werden diskutiert.

АНАЛИЗ ПОТЕНЦИАЛА И ФУНКЦИИ ТОКА ДВУМЕРНЫХ СТАЦИОНАРНЫХ КОНВЕКТИВНЫХ ДИФФУЗИОННЫХ УРАВНЕНИЙ, ВКЛЮЧАЮЩИХ ПОЛЯ ЛАПЛАСА

Аннотация—Дана методика решения стационарного двумерного конвективного диффузионного уравнения, включающего поля внешней силы с потенциалами Лапласа. Методика основана на возможности отыскания сопряженной потенциалу функции тока. Эти функции, таким образом, могут рассматриваться как независимые координаты. Показано, что уравнение конвективной диффузии может быть преобразовано в уравнение, в котором независимыми переменными являются потенциал и функция тока. В полученном уравнении переменные разделяются. Представлены две иллюстративные задачи. На примере 'псевдодвумерной' задачи показана плодотворность метода даже в том случае, когда удовлетворяются не все требования.